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Translated by M.D.F.

PMM U.S.S.R., Vol.51, No.3, pp.381-385, 1987
Printed in Great Britain

0021-8928/87 $10.00+0.00

ON OPTIMAL PLASTIC ANISOTROPY*

N.V. BANICHUK and V.V. KOBELEV

An approach is developed for the optimal design of a structure, based on optimization of the anisotropic plastic properties of materials. Problems of maximizing the ultimate plastic rupture load as a result of optimal orientation of the plastic anisotropy axes in the structure elements are formulated. Necessary conditions are presented for optimality in the three-dimensional problem of the theory of ultimate plastic equilibrium. Cases of the torsion and bending of plastic rods are considered. The bilateral achievable estimates of the ultimate loads are obtained. It is noted that the conditions for achieving the upper and lower bounds agree with the necessary optimality conditions. It is proved that the maximum ultimate load is realized in the case when the direction with the greatest yield point of the material agrees with the direction given by the tangential stress vector at the time of exhaustion of the carrying capacity.

1. Formulation of the problem. Optimality conditions. We consider a deformable body that occupies a domain Ω with boundary Γ. The body material is considered to be ideally elastic-plastic. The flow state occurs at a certain point if the flow condition

$$g(\sigma_{ij}, k) \leqslant 0 \tag{1.1}$$

is satisfied with the equality sign ($g = 0$). If $g < 0$ then the material behaves elastically. Here k is the plasticity constant, g is a given function, and σ_{ij} are stress tensor components The equation $g(\sigma_{ij}, k) = 0$ in the stress space yields a family of convex surfaces enclosing the origin. It is assumed in the problems studied below that flow domains occur when loads are applied to the body. The very appearance of flow domains is considered allowable, however, it is required that the plastic strains should not result in exhaustion of the carrying capacity and to body rupture. Exhaustion of the carrying capacity is understood to be unbounded growth of strains under constant loads (/1, 2/).

To estimate the carrying capacity, the theorem on ultimate equilibrium is used, according to which the body sustains applied loads if a safe statically possible field of stresses σ_{ij} exists, i.e., a stress distribution satisfying the equilibrium equations and boundary conditions

$$\sigma_{ij,j} + q_i = 0, \ (\sigma_{ij}n_j)_{\Gamma_\sigma} = T_i \tag{1.2}$$

and such that

$$g(\sigma_{ij}, k) < 0 \tag{1.3}$$

Here n_i are components of the unit external normal vector to the body surface $(n_in_i = 1)$, and Γ_σ is the part of the body surface on which the loads T_i are given. On the rest of the

**Prikl.Matem.Mekhan.*, 51, 3, 489-495, 1987

surface the body is considered clamped, i.e., it is considered that the displacements are zero. The Roman subscripts in this section take the values 1, 2, 3 and the values 1 and 2 in Sects.2 and 3. Summation is over repeated subscripts and the subscript after the comma denotes differentiation with respect to the corresponding coordinate.

The problem of maximization because of appropriate selection of the plastic anisotropy axes orientation at each point of the body can be formulated by using the theorem presented above /3/.

At each point of the body let the direction of the plastic anisotropy axes be given by the unit vectors $\mathbf{j}_1, \mathbf{j}_2, \mathbf{j}_3$; $\mathbf{i}_1, \mathbf{i}_2, \mathbf{i}_3$ are the directions of a global Cartesian coordinate system. Orientation of the plastic anisotropy axes relative to the axes of the global coordinate system is given by the magnitudes of the cosines of the angles between the directions $\mathbf{j}_l$ and $\mathbf{i}_k$, i.e., by the quantities $m_{lk} = \mathbf{j}_l \cdot \mathbf{i}_k$. The matrix of the cosines is orthogonal: $m_{ij}m_{jk} = \delta_{ik}$, where δ_{ik} is the Kronecker delta. Since the orientation varies from point to point, the magnitudes of the direction cosines are functions of the coordinates. The optimization problem consists of seeking the orientation of the anisotropy axes, i.e., the functions $m_{lk}(x_i)$, from the condition for the maximum of the ultimate load maximum

$$p_* = \max_{m_{ij}} p(m_{ij}) \tag{1.4}$$

upon compliance with the condition (1.1), where in (1.2)

$$q_i = pq_i^\circ, \ T_i = pT_i^\circ \tag{1.5}$$

and T_i°, q_i° are given functions. Unlike (1.3), inequality (1.1) is not strict. Let us give this substitution a foundation. We note first that the set of values of σ_{ij} satisfying conditions (1.2) and (1.3) is not closed. This circumstance makes an incorrect formulation of the optimization problem (1.2)-(1.4).

We will use the following natural method for regularization. We consider two nearby flow surfaces corresponding to the initial parameter k and differing only slightly in the magnitude of the parameter k_ε, $k > k_\varepsilon$. The difference $k - k_\varepsilon = \varepsilon > 0$ can be considered as small as desired. By using geometric representations about the flow surfaces it can be noted that for statically allowable stress fields satisfying the inequality $g(\sigma_{ij}, k_\varepsilon) \leqslant 0$ and (1.2), satisfaction of the strict inequality (1.3) would also be ensured. Consequently, such fields are safe statically allowable fields for the original flow surface.

The solution of the optimization problem can be determined as accurately as desired by giving a sufficiently small value of ε and replacing condition (1.3) by (1.1) in (1.2)-(1.3). Application of this method, based on the introduction of a modified flow surface, is also justified from the viewpoint that in practice the constant k is always determined with a certain error. Later we assume a plasticity criterion in the form of a quadratic function of the stresses /4/

$$g(\sigma_{ij}, k) = \sigma_{ij}B_{ijkl}\sigma_{kl} - k \leqslant 0, \quad B_{ijkl} = B_{klij} = B_{jikl}$$

Let us derive the optimality conditions for problem (1.1), (1.4) and (1.5). We will write the relation between the components of the tensor B_{ijkl} and the plastic constants tensor in the principal axes b_{ijkl}, as well as the expression for the variations δB_{ijkl} due to the variation δm_{ij} of the cosines m_{ij}

$$B_{ijkl} = b_{pqns}m_{ip}m_{jq}m_{kn}m_{ls} \tag{1.6}$$
$$\delta B_{ijkl} = 4B_{ijks}m_{sp}\delta m_{pl}$$

Representing the functional being optimized in the form

$$p = \frac{1}{\operatorname{mes}\Omega}\int_\Omega p\, d\Omega, \quad p_{,i} \equiv 0 \tag{1.7}$$

and taking account of (1.1), (1.4), (1.5) and (1.7), we form the Lagrange functional

$$J = \frac{1}{\operatorname{mes}\Omega}\int_\Omega p\, d\Omega + \int_\Omega \psi_i(\sigma_{ij,j} + q_i)\, d\Omega + \int_\Omega \lambda(g + \mu^2)\, d\Omega + \int_\Omega \eta_{ij}(m_{ik}m_{kj} - \delta_{ij})\, d\Omega \tag{1.8}$$

To obtain the variations of the Lagrange functional (1.8), we vary the tensor B_{ijkl}. We use the relationship (1.6), the boundary conditions from (1.5) and we take into account that the variations of p in (1.8) are independent of the space coordinates since $p_{,i} = 0$. We define the conjugate variables ψ_i as functions satisfying the following differential equations and boundary conditions

$$^1/_2(\psi_{i,j}+\psi_{j,i})=\lambda\frac{\partial g}{\partial\sigma_{ij}}\equiv 2\lambda B_{ijkl}\sigma_{kl} \qquad (1.9)$$
$$(\psi_i)_{\Gamma_u}=0,\quad \Gamma=\Gamma_\sigma\cup\Gamma_u$$

Moreover, we assume that λ and μ are subject to additional non-stiffness conditions

$$\lambda\mu=0 \quad \text{in} \quad \Omega \qquad (1.10)$$

Using (1.9)-(1.10) and the requirement $\delta J=0$, we obtain the optimality condition in the form

$$\int_\Omega(\lambda\sigma_{ij}B_{ijkl}\sigma_{ls}+\eta_{ks})\,m_{sp}\delta m_{pk}\,d\Omega=0$$

which by virtue of the symmetry $\eta_{ks}=\eta_{sk}$ and arbitrariness of δm_{pk} reduces to the relationship

$$\sigma_{ij}B_{ijkl}\sigma_{ls}=\sigma_{ij}B_{ijsl}\sigma_{lk} \qquad (1.11)$$

If $\mathbf{B}$ is the plastic constants tensor with components B_{ijkl}, and $\boldsymbol\sigma$ is the stress tensor, then the last equality is written in the symmetric form

$$\boldsymbol\sigma\cdot\cdot\mathbf{B}\cdot\boldsymbol\sigma=\boldsymbol\sigma\cdot\mathbf{B}\cdot\cdot\boldsymbol\sigma \qquad (1.12)$$

The one and two dots between the symbols in (1.12) denote the simple and double scalar product.

2. Optimal orientation of the plastic anisotropy axes in a twisted rod.

Let us consider the torsion of a cylindrical rod subjected to moments applied to its ends. We assume that the transverse section of the rod S is a simply-connected domain. The x_3 axis of the $x_1x_2x_3$ orthogonal coordinate system is oriented in the direction of the cylinder generator, while the x_1 and x_2 axes in the plane of the transverse section. The lateral sides of the rod are load-free. It is also assumed that the material is continuously inhomogeneous and has a plane of elastic symmetry normal to the generator (the x_3 axis) at each point. The material is orthotropic at each point, where the two orthotropy axes are in the plane of the transverse section and their orientation is independent of the coordinate x_3. Under torsion the two stress tensor components $\tau_{13}(x_1,x_2)$, $\tau_{23}(x_1,x_2)$ ($\tau_1=\sigma_{13}$, $\tau_2=\sigma_{23}$) are different from zero. The torque is expressed in terms of the stress tensor components as follows:

$$M=\int_S(\tau_2x_1-\tau_1x_2)\,dS \qquad (2.1)$$

We write the equilibrium equation and boundary conditions

$$\tau_{1,1}+\tau_{2,2}=0 \quad \text{on} \quad S \qquad (2.2)$$
$$\tau_1n_1+\tau_2n_2=0 \quad \text{on} \quad \partial S$$

(∂S is the boundary of the domain S). Since only two stress tensor components are not zero, the plasticity condition is rewritten in the abbreviated form

$$B_{ij}\tau_i\tau_j\leqslant k \quad (B_{11}=B_{1313},\ B_{12}=B_{1323},\ B_{22}=B_{2323}) \qquad (2.3)$$

The notation (indicated in the parentheses) for the anisotropy parameters calculated in the global coordinate system $x_1x_2x_3$ with directions $\mathbf{i}_1$, $\mathbf{i}_2$, $\mathbf{i}_3$ is introduced here. These quantities are functions of the coordinates and depend on the orientation of the anisotropy axes in the plane of the section.

In addition to the $x_1x_2x_3$ coordinate system, we introduce a local $y_1y_2y_3$ coordinate system whose directions $\mathbf{j}_1$, $\mathbf{j}_2$, $\mathbf{j}_3$ agree at each point with the direction of the anisotropy axes of the material. The x_3 axis is parallel to the y_3 axis and the angle between the y_1 and x_1 axes is denoted by $\alpha(x_1,x_2)$. The flow condition in the $y_1y_2y_3$ coordinate system is also a quadratic form of the stress, where the parameters b_{ij} characterizing the running state of the anisotropy in the $y_1y_2y_3$ coordinate system are independent of the point coordinates and are constants. These quantities are expressed in terms of the flow limits τ_1°, τ_2° for a shift with respect to the principal anisotropy axes $b_{11}=1/\tau_1^\circ$, $b_{22}=1/\tau_2^\circ$, $b_{12}=0$ ($\tau_1^\circ\geqslant\tau_2^\circ$). Note that the coefficients b_{ij} are components of a symmetric positive-definite matrix. If the rotation matrix $m(\alpha)$ with components $m_{11}=m_{22}=\cos\alpha$, $m_{12}=-m_{21}=\sin\alpha$ is introduced, then the relation between the coefficients B_{ij} and b_{ij} can be represented in the form

$$B_{ij}=m_{ik}m_{jl}b_{kl}$$

We express the stress tensor components in terms of the stress function $\Psi(x_1,x_2)$

$$\tau_1=\Psi_{,2},\quad \tau_2=-\Psi_{,1} \qquad (2.4)$$

As is known, the equilibrium equations are satisfied identically when (2.4) is substituted into (2.2) and the boundary condition reduces to the function Ψ being zero on the boundary.

The plasticity condition (2.3) is rewritten in the form

$$A_{ij}\Psi_{,i}\Psi_{,j} = \nabla\Psi\cdot\mathbf{A}\cdot\nabla\Psi \leqslant k \qquad (2.5)$$
$$\mathbf{A} = \mathbf{B}^{-1}|\mathbf{B}| = |\mathbf{b}|(\mathbf{m}\cdot\mathbf{b}\cdot\mathbf{m}^*)^{-1}, \quad \mathbf{m}\cdot\mathbf{m}^* = \mathbf{E}$$

The ultimate torque equals the extremal value of the functional

$$M = 2\max_{\Psi\in D}\int_S \Psi\, dS \qquad (2.6)$$

in the set

$$D = \{\Psi:\ \nabla\Psi\cdot\mathbf{A}\cdot\nabla\Psi \leqslant k,\ \Psi|_{\partial S} = 0\}$$

If the plasticity function Ψ belongs to the set D, then the equilibrium and plasticity conditions are satisfied. The optimization problem formulated in Sect.1 for the three-dimensional problem becomes the following for the specific case of the state of stress: determine the optimal orientation of the anisotropy axis from the condition for the maximum of the ultimate torque

$$\max_{\alpha(x_1,\, x_2)} M \qquad (2.7)$$

The necessary optimality conditions can be obtained directly from (1.12). In the notation used here these equations are rewritten in the form

$$(B_{11}\tau_1 + B_{12}\tau_2)\,\tau_2 = (B_{12}\tau_1 + B_{22}\tau_2)\,\tau_1$$

and are satisfied identically if

$$B_{11}\tau_1 + B_{12}\tau_2 = \Lambda\tau_1, \quad B_{12}\tau_1 + B_{22}\tau_2 = \Lambda\tau_2 \qquad (2.8)$$

Relying on relationships (2.4) and (2.5) we convert (2.8) to the form

$$\mathbf{A}\cdot\nabla\Psi = \lambda\nabla\Psi,\ \lambda = \Lambda\ |\mathbf{B}| \qquad (2.9)$$

($|\mathbf{B}|$ is the determinant of the matrix $\mathbf{B}$).

Later the mechanical meaning of (2.9) will be clarified.

Let us show that the maximum M of the functional is achieved when a regime corresponding to a large eigenvalue A is realized in the whole domain S. We introduce the following quantities into the analysis

$$M^* = 2\max_{\Psi\in D^*}\int_S \Psi\, dS, \quad M^{**} = 2\max_{\Psi\in D^{**}}\int_S \Psi\, dS \qquad (2.10)$$
$$D^* = \{\Psi:\ \lambda_{\min}(\mathbf{b})\ \nabla\Psi\cdot\nabla\Psi \leqslant k,\ \Psi|_{\partial S} = 0\}$$
$$D^{**} = \{\Psi:\ \lambda_{\max}(\mathbf{b})\ \nabla\Psi\cdot\nabla\Psi \leqslant k,\ \Psi|_{\partial S} = 0\}$$

It can be shown that

$$M^{**} \leqslant M \leqslant M^* \qquad (2.11)$$

where the lower boundary for $M\ (\inf M = M^{**})$ is achieved when the equality $\mathbf{A}\cdot\nabla\Psi = \lambda_{\max}(\mathbf{b})\ \nabla\Psi$ is satisfied in the whole domain, and the upper boundary $(\sup M = M^*)$ is realized for $\mathbf{A}\cdot\nabla\Psi = \lambda_{\min}(\mathbf{b})\nabla\Psi$.

Indeed, by the efinition of the maximum and minimum eigenvalue

$$\lambda_{\min}(\mathbf{b})\ \nabla\Psi\cdot\nabla\Psi \leqslant \nabla\Psi\cdot\mathbf{A}\cdot\nabla\Psi \leqslant \lambda_{\max}(\mathbf{b})\ \nabla\Psi\cdot\nabla\Psi \qquad (2.12)$$

The upper inequality is given a foundation by the definition of the eigenvalues

$$\lambda_{\max}(\mathbf{A}) = \max_u \frac{u\cdot\mathbf{A}\cdot u}{u\cdot u}, \quad \lambda_{\min}(\mathbf{A}) = \min_u \frac{u\cdot\mathbf{A}\cdot u}{u\cdot u}$$

and the chain of equalities $\lambda_{\max}(\mathbf{A}) = |\mathbf{B}|\lambda_{\min}(\mathbf{B}) = |\mathbf{b}|\lambda_{\min}(\mathbf{b}) = \lambda_{\max}(\mathbf{b})$, since $\lambda_{\min}(\mathbf{b})\lambda_{\max}(\mathbf{b}) = |\mathbf{b}|$.

From (2.10) and (2.12) it follows that $D^{**} \subset D \subset D^*$ and, consequently, inequlity (2.11) is valid.

It therefore follows that the ultimate moment reaches a maximum when the directions with the highest yield at each point are in agreement with the direction of action of the maximum tangential stress. The torsion equation reduces to an equation describing the plastic torsion of isotropic rods with the yield point $k\cdot\lambda_{\min}^{-1}(\mathbf{b})$.

The theory of ultimate plastic torsion of isotropic rods has been well developed /5, 6/. The function Ψ is mapped by a ruled surface constructed on the contour, whose rectilinear generators are normal to the contour line ∂S and have a constant slope $k\lambda_{\min}^{-1}(\mathbf{b})$ to the plane of the transverse section. This is the so-called surface of equal slope (SES). The direction of the maximal shear stress (and therefore of the maximal yield point in an optimal rod) agrees with the SES level lines. The slope of the SES in an optimal rod is proportional to the highest yield point. The magnitude of the ultimate moment equals twice the volume included under the SES. Therefore, the geometric meaning of the inequalities (2.11) is evident; they express the fact that the surface corresponding to the plasticity function Ψ is included between two SES for arbitrary orientation of the anisotropy axes.

3. Optimal orientation of the anisotropy axes for bending of a cantilever. The state of stress is examined in a prismatic cantilever rod loaded by surface forces distributed over its unclamped endface. We superpose the origin on the stiffly framed left endface of the rod. Let the side surface of the rod be force-free, and P the principal vector of the surface forces on the right end $(x_3 = l)$ directed along the x_1 axis. The surface loads on the endface produce a moment M relative to the centre of inertia whose vector is directed along x_2. Therefore, a transverse force P and a bending moment M act on the rod. The stress tensor components τ_1, τ_2 and σ_{33} differ from zero at each section of the rod. The normal stress σ_{33} is represented by the formula

$$\sigma_{33} = -I^{-1}[M + P(l - x_3)]x_1 \tag{3.1}$$

and the statics equations in the volume and on the surface are written in the form

$$\tau_{1,1} + \tau_{2,2} = -I^{-1}Px_1 \qquad (3.2)$$
$$\tau_{1,3} = 0,\ \tau_{2,3} = 0,\ \tau_1 n_1 + \tau_2 n_2 = 0$$

(I is the moment of inertia of a section relative to the x_2 axis). It is assumed that the transverse section has an axis of symmetry that agrees with the force line of action. An S.P. Timoshenko stress function $\Psi(x_1, x_2)$ is introduced in terms of which the shear stresses are expressed.

$$\tau_1 = \Psi_{,2} - {}^1/_2 I^{-1}Px_1^2 + G(x_2)$$
$$\tau_2 = -I^{-1}P\Psi_{,1}$$

The boundary condition (3.2) is rewritten by using the stress function as

$$\partial\Psi/\partial s = [{}^1/_2 I^{-1}Px_1^2 - G(x_2)]\partial y/\partial s$$

where $\partial/\partial s$ is the derivative with respect to the contour. If the function $G(x_2)$ is selected in such a way that the condition $G(x_2) = {}^1/_2 I^{-1}Px_1^2$ is satisfied on the contour then the boundary condition takes the form $\Psi = 0$.

We will consider the plane of the cantilever transverse section to be the plane of plastic symmetry. Under this assumption the plasticity condition is written in the form

$$B_{ij}\tau_i\tau_j + B_{3333}\sigma_{33}^2 \leqslant k \tag{3.3}$$

We assume that the bending moment $M = -\int \sigma_{33}x_1 ds$ and the transverse force $P = \int \tau_1 ds$ vary in proportion to one parameter. Under such a proportional loading the bending moment is a linear function the the transverse force $M = \varkappa P$ ($\varkappa$ is a quantity dependent only on the location of the section on the rod axis). If a stress function $\psi(x_1, x_2)$ exists such that the stresses calculated with its aid satisfy inequality (3.3) at each point, then the effective loads do not reach the limit values.

The optimization problem consists of seeking a distribution of the orientation of the plastic anisotropy axes such that the limit load reaches the maximum value.

It can be shown that the optimal orientation of the anisotropy axes is such that the direction of the shear stress vector $\{\tau_1, \tau_2\}$ acting in the plane of the rod transverse section will agree with the direction of the eigenvector of the matrix $\mathbf{B}$ corresponding to the smallest eigenvalue. Indeed such an orientation corresponds to the minimum value of the quadratic form on the left-hand side of (3.3).

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Translated by M.D.F.